A theorem about areas of a quadrilateral and a triangle formed by mid-points of its sides

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Theorem 1. *The triangle formed by the mid-points of any three sides of a quadrilateral has one-fourth area of the quadrilateral.*

1 Area of a triangle

Area of a trangle whose three vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by

$$A = \frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right| \tag{1}$$

We use this formula for the rest of the results.

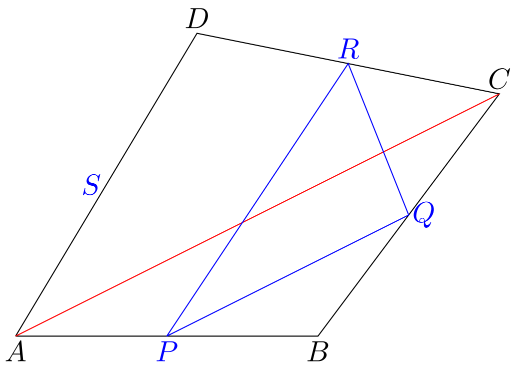
2 Area of the quadrilateral

Let us consider a quadrilateral whose vertices are

$$\begin{aligned} A &\Rightarrow (x_1, y_1) \\ B &\Rightarrow (x_2, y_2) \\ C &\Rightarrow (x_3, y_3) \\ D &\Rightarrow (x_4, y_4) \end{aligned}$$

Let P, Q, R, S be the midpoints of $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$. So,

$$P \Rightarrow \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$Q \Rightarrow \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$
$$R \Rightarrow \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2}\right)$$
$$S \Rightarrow \left(\frac{x_4 + x_1}{2}, \frac{y_4 + y_1}{2}\right)$$



The area of the quadrilateral $ABCD$ is the sum of the areas of triangles ABC and ACD.

$$\begin{aligned}
A_{ABC} &= \frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right| \\
A_{ACD} &= \frac{1}{2} \left| x_1(y_3 - y_4) + x_3(y_4 - y_1) + x_4(y_1 - y_3) \right| \\
A_{ABCD} &= \frac{1}{2} \left| x_1(y_2 - y_3 + y_3 - y_4) + x_2(y_3 - y_1) + x_3(y_1 - y_2 + y_4 - y_1) + x_4(y_1 - y_3) \right| \\
&= \frac{1}{2} \left| x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3) \right| \qquad (2)
\end{aligned}$$

3 Area of the triangle formed by three midpoints

WLOG, let us consider the triangle formed by P, Q and R.

$$
\begin{aligned}
A_{PQR} &= \frac{1}{2}| \\
&\quad \frac{x_1+x_2}{2}\cdot\left(\frac{y_2+y_3}{2}-\frac{y_3+y_4}{2}\right)+ \\
&\quad \frac{x_2+x_3}{2}\cdot\left(\frac{y_3+y_4}{2}-\frac{y_1+y_2}{2}\right)+ \\
&\quad \frac{x_3+x_4}{2}\cdot\left(\frac{y_1+y_2}{2}-\frac{y_2+y_3}{2}\right) \\
&\quad | \\
&= \frac{1}{8}| \\
&\quad (x_1+x_2)(y_2+y_3-y_3-y_4)+ \\
&\quad (x_2+x_3)(y_3+y_4-y_1-y_2)+ \\
&\quad (x_3+x_4)(y_1+y_2-y_2-y_3) \\
&\quad | \\
&= \frac{1}{8}\left|(x_1+x_2)(y_2-y_4)+(x_2+x_3)(y_3+y_4-y_1-y_2)+(x_3+x_4)(y_1-y_3)\right| \\
&= \frac{1}{8}|x_1y_2-x_1y_4+x_2y_2-x_2y_4+x_2y_3+x_2y_4-x_2y_1-x_2y_2 \\
&\quad +x_3y_3+x_3y_4-x_3y_1-x_3y_2+x_3y_1-x_3y_3+x_4y_1-x_4y_3| \\
&= \frac{1}{8}|x_1y_2-x_1y_4-x_2y_1-x_2y_2+x_2y_2+x_2y_3-x_2y_4+x_2y_4 \\
&\quad +x_3y_1-x_3y_1-x_3y_2+x_3y_3-x_3y_3+x_3y_4+x_4y_1-x_4y_3| \\
&= \frac{1}{8}|x_1y_2-x_1y_4-x_2y_1+x_2y_3-x_3y_2+x_3y_4+x_4y_1-x_4y_3| \\
&= \frac{1}{8}|x_1(y_2-y_4)+x_2(y_3-y_1)+x_3(y_4-y_2)+x_4(y_1-y_3)| \qquad (3)
\end{aligned}
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From (2) and (3),

$$A_{PQR} = \frac{1}{4}\cdot A_{ABCD}$$

Hence the result. □